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LETTER TO THE EDITOR

Generalized Wigner surmise for (2×2) random matrices

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Abstract
We present new analytical results concerning the spectral distributions for (2×2) random real symmetric matrices which generalize the Wigner surmise.

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1. Introduction

The level statistics of a quantum system represents the most significant, although not the only [1], signature of quantum chaos. The Poisson and Wigner distributions of dimensionless nearest-neighbour spacing, s,

$$\tilde{p}_P(s) = \exp(-s), \tag{1}$$

$$\tilde{p}_W(s) = \frac{\pi s}{2} \exp\left(-\frac{\pi s^2}{4}\right), \tag{2}$$

are known in quantum chaos theory as two universalities that correspond to two extreme cases of classical dynamics, namely purely regular and completely chaotic (see, e.g., [2]). The majority of many-body systems such as nuclei, molecules, atoms or solids (see [1, 3, 4] and references therein) have been found to be chaotic although for such complex systems no classical limit can be constructed.

As was first recognized by Wigner [5], the nearest-neighbour spacing distribution (NNSD) (2) corresponds well to the eigenvalue distributions of random matrices, and this explains the importance of the random matrix theory [6] for studying statistical properties of many-body systems. Particular attention was paid to Gaussian ensembles. In fact, assuming that (i) the elements of the Hamiltonian matrix are *independent real variables* and (ii) the matrix distribution is *invariant* under an *orthogonal* transformation of the basis states (see, e.g., chapter 3 in [6]), one finds that the matrix elements are independent Gaussian variables with zero mean and with variance satisfying the conditions $\sigma_{ij}^2 = (1+\delta_{ij})\sigma^2$. Imposing particular symmetries on the Hamiltonian, one gets [7] Gaussian orthogonal, unitary or symplectic ensembles (GOE, GUE or GSE, respectively), which are widely and successfully applied in many fields of physics (see, e.g., reviews [1, 3, 4]).

Up to now, excited atomic nuclei have been considered to be the best examples of chaotic quantum systems [1]. Starting from slow-neutron scattering experiments, which were first described in terms of random matrices [5], much nuclear structure data has been analysed in the context of chaos. However, it was repeatedly noticed that the experimental data do not exactly match the distribution (2) and exhibit slight deviations [8, 9]. These deviations are thought to be caused by the fact that the real system is not purely chaotic, but can be the quantum analogue of a classical system that is transitional between chaotic and integrable. In this context, a few phenomenological formulae were proposed and analysed. The most famous are the Brody [8] distribution,

$$\tilde{p}_\omega(s) = (\omega+1)\alpha s^\omega \exp\left(-\alpha s^{\omega+1}\right), \qquad \alpha = \left[\Gamma\left(\frac{\omega+2}{\omega+1}\right)\right]^{\omega+1}, \tag{3}$$

which was shown to match better the experimental data [9] on both high- and low-energy nuclear spectra, and the Berry–Robnik distribution [10]

$$\tilde{p}_{\rm BR}(s) = \exp[(q-1)s]\{(1-q)^2 \mathrm{erfc}(\sqrt{\pi} q s/2) + [2q(1-q) + (\pi/2)q^3 s] \exp[-(\pi/4)q^2 s^2]\}. \tag{4}$$

Although the Brody distribution takes the Poisson form for $\omega = 0$ and the Wigner form for $\omega = 1$, it has the unrealistic property that its derivative goes to infinity at $\varepsilon = 0$ [11]. Moreover, the parameter ω has no clear physical meaning. On the other hand, the distribution (4) does not give a level repulsion for non-integrable systems. Caurier *et al* [11] considered a model which allowed them to simulate the transition from integrability to chaos and succeeded in deriving the asymptotic limits for small and large neighbour spacings in the near-integrable limit.

The idea of deriving a third universality class corresponding to intermediate statistics has been actively pursued in recent years. In particular, in [12, 13], the distribution of n nearest neighbours,

$$\tilde{p}^{(\beta)}(n,s) = \frac{(\beta+1)^{n(\beta+1)}}{\Gamma[n(\beta+1)]} s^{[n(\beta+1)-1]} \exp\left[-(\beta+1)s\right], \tag{5}$$

was shown to be relevant to a certain class of exactly solvable models with nearest and next-to-nearest neighbour interactions, generalizing the results obtained earlier [14, 15] on pseudointegrable billiards and the short-range Dyson models. Remarkably, the NNSD given by (5) with $n = 1$ exhibits a level repulsion $\sim s^\beta$ and falls to zero at s as $\exp\left[-(\beta+1)s\right]$. For $\beta = 1$ it is referred to as the semi-Poisson distribution [14, 15].

In this letter we argue that one might search for an explanation of the discrepancy between data and random matrix theory simply by generalising the random matrix ensemble used. In fact, the Gaussian ensemble is defined by the two assumptions (i) and (ii) mentioned above, and their applicability should be carefully checked for a given physical system. As far as nuclear physics is concerned, the deviations of the experimental data on slow-neutron or (p, p′) resonances from a random matrix description could simply arise from the *non-invariance* of the random matrix ensemble under an orthogonal transformation of a basis; i.e., the assumption (ii) is violated.

In addition, it is well known that realistic interactions in many-body nuclear, molecular or atomic systems are predominantly of one- and two-body nature, implying that the distribution of the matrix is not only *not invariant* under an orthogonal (unitary) transformation of the basis, but also that the elements of the Hamiltonian matrix are *not independent* (if the number of particles is more than two); i.e., both assumptions for a Gaussian ensemble no longer hold.

In this context, French and Wong [16] and Bohigas and Flores [17] independently introduced the two-body random ensemble (TBRE), which is characterized by a Gaussian level density distribution, rather than a semi-circle provided by a GOE [16]. The level distributions of experimental nuclear spectra confirmed this result. The NNSD relevant for a TBRE was found numerically to be fairly well represented by the Wigner surmise [18], although recently it has been pointed out [19] that the NNSD given by (5) with $n = 1$ and a certain real value of β fits better the shell-model spectrum obtained with the sd interaction of Wildenthal [20].

Given the importance and actuality of these investigations, we present in this letter some analytical results concerning the properties of (2×2) random symmetric matrices for which the assumption (ii) mentioned above is not satisfied. First, we derive the Hamiltonian distribution as a function of its eigenvalues and we calculate the NNSD which *generalizes* the well-known Wigner surmise [5]. We show that the model allows us to describe the transition from purely chaotic to asymptotically nearly integrable limits, being different from the intermediate statistics mentioned above. Then, for a particular case, we give the analytical expressions for the lowest moments of this distribution. Finally, we propose a method to derive the moments of the eigenvalue distribution without knowledge of an explicit expression for the distribution.

2. Generalized Wigner surmise

Let us consider a (2×2) real symmetric matrix

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}, \tag{6}$$

whose elements are independent Gaussian variables with zero mean and variance σ_{ij}^2, and $H_{12} = H_{21}$. The probability density of the matrix H is then given by

$$p(H) = \frac{1}{(2\pi)^{3/2}\sqrt{\sigma_{11}^2\sigma_{12}^2\sigma_{22}^2}} \exp\left[-\left(\frac{H_{11}^2}{2\sigma_{11}^2} + \frac{H_{12}^2}{2\sigma_{12}^2} + \frac{H_{22}^2}{2\sigma_{22}^2}\right)\right]. \tag{7}$$

Each matrix H can be diagonalized in an orthogonal basis and therefore $H = O^t D O$, with

$$O = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} E_\alpha & 0 \\ 0 & E_\beta \end{pmatrix}.$$

Similar to the case of the GOE [21], we find that in the general case

$$\begin{aligned} H_{11} &= E_\alpha \cos^2\theta + E_\beta \sin^2\theta, \\ H_{12} &= \left(E_\alpha - E_\beta\right)\cos\theta\sin\theta, \\ H_{22} &= E_\alpha \sin^2\theta + E_\beta \cos^2\theta. \end{aligned}$$

We deduce that the probability density expressed in terms of the eigenvalues and the angle θ is

$$\begin{aligned} p\left(E_\alpha, E_\beta, \theta\right) = {} & \frac{E_\alpha - E_\beta}{(2\pi)^{3/2}\sqrt{\sigma_{11}^2\sigma_{22}^2\sigma_{12}^2}} \\ & \times \exp\left\{-\frac{\left[E_\alpha\Sigma^2 - (E_\alpha - E_\beta)\left(\sigma_{11}^2\cos^2\theta + \sigma_{22}^2\sin^2\theta\right)\right]^2}{2\sigma_{11}^2\sigma_{22}^2\Sigma^2}\right\} \\ & \times \exp\left[-\frac{1}{2}(E_\alpha - E_\beta)^2\left(\frac{\cos^2(2\theta)}{\Sigma^2} + \frac{\sin^2(2\theta)}{4\sigma_{12}^2}\right)\right] \end{aligned} \tag{8}$$

where $\Sigma^2 = \sigma_{11}^2 + \sigma_{22}^2$ and $E_\alpha - E_\beta \geqslant 0$.

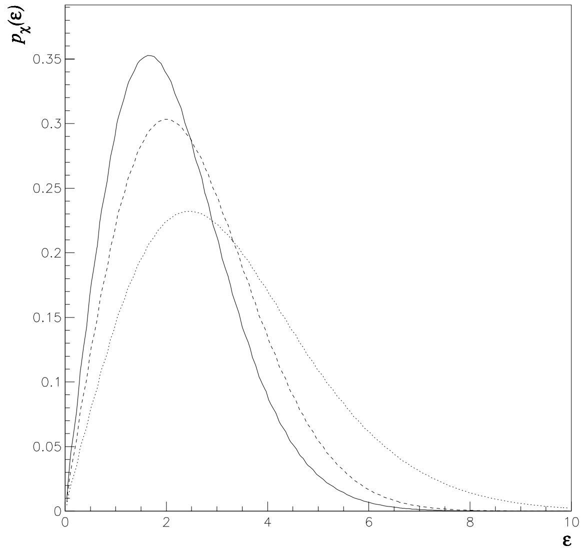

Figure 1. Quadratic Rayleigh–Rice distributions for $\chi = 1$ (solid line), $\chi = 2$ (dashed line) and $\chi = 5$ (dotted line) with $\sigma^2 = 1$.

The nearest-neighbour spacing distribution for the variable $\varepsilon = E_\alpha - E_\beta = Ds$, with the mean spacing $D = \int \varepsilon \tilde{p}(\varepsilon)\, \mathrm{d}\varepsilon$, is given by the following integral

$$\tilde{p}\,(\varepsilon) = \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \int_{-\infty}^{\infty} \mathrm{d}E_\alpha \int_{-\infty}^{E_\alpha} \mathrm{d}E_\beta \; p\left(E_\alpha, E_\beta, \theta\right) \, \delta\left(\varepsilon - E_\alpha + E_\beta\right), \tag{9}$$

from which we obtain

$$\tilde{p}\,(\varepsilon) = \frac{\varepsilon}{2\sqrt{\Sigma^2 \sigma_{12}^2}} \exp\left[-\frac{\varepsilon^2\left(\Sigma^2 + 4\sigma_{12}^2\right)}{16\Sigma^2\sigma_{12}^2}\right] I_0\left(\frac{\varepsilon^2\left(\Sigma^2 - 4\sigma_{12}^2\right)}{16\Sigma^2\sigma_{12}^2}\right) \tag{10}$$

where I_0 is a modified Bessel function of the first kind.

The expression (10) looks like a Rayleigh–Rice distribution, well known in signal theory [22], except for the argument of I_0, which is not linear as in the usual Rayleigh–Rice distribution but quadratic. This is why we will refer to $\tilde{p}\,(\varepsilon)$ as to a *quadratic* Rayleigh–Rice distribution.

Let us consider a particular case when the diagonal matrix elements have the same variance $\sigma_{11}^2 = \sigma_{22}^2$, which is χ times larger than the variance, $\sigma_{12}^2 = \sigma^2$, of the non-diagonal matrix elements, i.e., $\chi = \sigma_{11}^2/\sigma_{12}^2$. Then the eigenvalue distribution (8) reduces to

$$p_\chi(E_\alpha, E_\beta, \theta) = \frac{E_\alpha - E_\beta}{(2\pi\sigma^2)^{3/2}\chi} \exp\left[-\frac{E_\alpha^2 + E_\beta^2 + \frac{1}{4}(E_\alpha - E_\beta)^2(\chi - 2)\sin^2(2\theta)}{2\chi\sigma^2}\right], \tag{11}$$

while for the nearest-neighbour spacing we get

$$\tilde{p}_\chi(\varepsilon) = \frac{\varepsilon}{\sqrt{2\chi}2\sigma^2} \exp\left(-\frac{(\chi + 2)\,\varepsilon^2}{16\chi\sigma^2}\right) I_0\left(\frac{(\chi - 2)\varepsilon^2}{16\chi\sigma^2}\right). \tag{12}$$

For $\chi = 2$, expression (12) reduces to the Wigner surmise. The distributions $\tilde{p}_\chi(\varepsilon)$ are plotted in figure 1 for $\chi = 1$, $\chi = 2$ and $\chi = 5$.

Table 1. The moments up to $n = 5$ of the quadratic Rayleigh–Rice distribution (12) for $\sigma^2 = 1$. For $n = 1$, one obtains the mean spacing D. F is the hypergeometric function [23].

	$M_n = \int_0^\infty \varepsilon^n \tilde{p}_\chi(\varepsilon)\,\mathrm{d}\varepsilon$	values for $\chi = 2$	for $\chi = 5$
$n = 0$	1	1	1
$n = 1$	$4\chi\sqrt{2\pi}(\chi+2)^{-3/2}F\left[\frac{3}{4},\frac{5}{4},1,\left(\frac{\chi-2}{\chi+2}\right)^2\right]$	$\sqrt{2\pi}$	$1.3\sqrt{2\pi}$
$n = 2$	$2(2+\chi)$	8	14
$n = 3$	$96\chi^2\sqrt{2\pi}(\chi+2)^{-5/2}F\left[\frac{5}{4},\frac{7}{4},1,\left(\frac{\chi-2}{\chi+2}\right)^2\right]$	$12\sqrt{2\pi}$	$28.7\sqrt{2\pi}$
$n = 4$	$4\left(12+4\chi+3\chi^2\right)$	128	428
$n = 5$	$3840\chi^3\sqrt{2\pi}(\chi+2)^{-7/2}F\left[\frac{7}{4},\frac{9}{4},1,\left(\frac{\chi-2}{\chi+2}\right)^2\right]$	$240\sqrt{2\pi}$	$1139.2\sqrt{2\pi}$

For small values of ε, the distribution (12) goes linearly to zero,

$$\tilde{p}_\chi(\varepsilon) \propto \frac{1}{\sqrt{2\chi}2\sigma^2}\varepsilon, \tag{13}$$

while for large ε

$$\tilde{p}_\chi(\varepsilon) \propto \begin{cases} \sqrt{\dfrac{2}{\mid \chi - 2 \mid \sigma^2}}\exp\left(-\dfrac{\varepsilon^2}{4\chi\sigma^2}\right) & \chi \neq 2 \\ \dfrac{\varepsilon}{4\sigma^2}\exp\left(-\dfrac{\varepsilon^2}{8\sigma^2}\right) & \chi = 2. \end{cases} \tag{14}$$

The asymptotic behaviour has a functional dependence on ε similar to that of the Wigner surmise; i.e., the NNSD (11) goes linearly to zero for $\varepsilon \to 0$ and it falls down according to $\exp(-\varepsilon^2)$ for $\varepsilon \to \infty$. However, as seen from (13), (14), the natural dependence on χ provides a certain scaling.

To calculate various statistical characteristics, it is often required to know certain moments of the distribution. Thus, we have derived analytical expressions for some moments of the distribution (12), and the lowest are given in table 1.

From (12) and the expression of the mean spacing, D (cf table 1), we have derived the distribution of the dimensionless nearest-neighbour spacing, $s = \varepsilon/D$:

$$q_\chi(s) = D\,\tilde{p}_\chi(sD)\,. \tag{15}$$

For $\chi = 2$, one finds the Wigner distribution (2). It can be shown that for χ and $\chi' = 4/\chi$, the two functions q_χ and $q_{\chi'}$ are equal.

The distributions q_χ for $\chi = 4$, $\chi = 30$, $\chi = 500$ and $\chi = 1000$ are plotted in figure 2 and compared with the Poisson (dashed), Wigner (dotted) and semi-Poisson (dashed-dotted) distributions. Note that the semi-Poisson distribution can be fairly well approximated for $\chi = 30$ although its asymptotic behaviour for large s is different. As follows from figure 2, the parameter χ allows us to describe a transition of a quantum system from a completely chaotic limit ($\chi = 2$) to a nearly integrable one ($\chi \to \infty$ or $\chi \to 0$). However, the integrability characterized by the Poisson distribution (1) can never be reached (compare this with the model of Caurier *et al* [11]).

Indeed, integrating p_χ in (11) over E_β from $-\infty$ to E_α and over E_α from $-\infty$ to $+\infty$, we obtain the angular distribution

$$r_\chi(\theta) = \frac{1}{\pi}\sqrt{\frac{\chi}{2}}\frac{1}{\left[1+\frac{1}{2}(\chi-2)\sin^2(2\theta)\right]}. \tag{16}$$

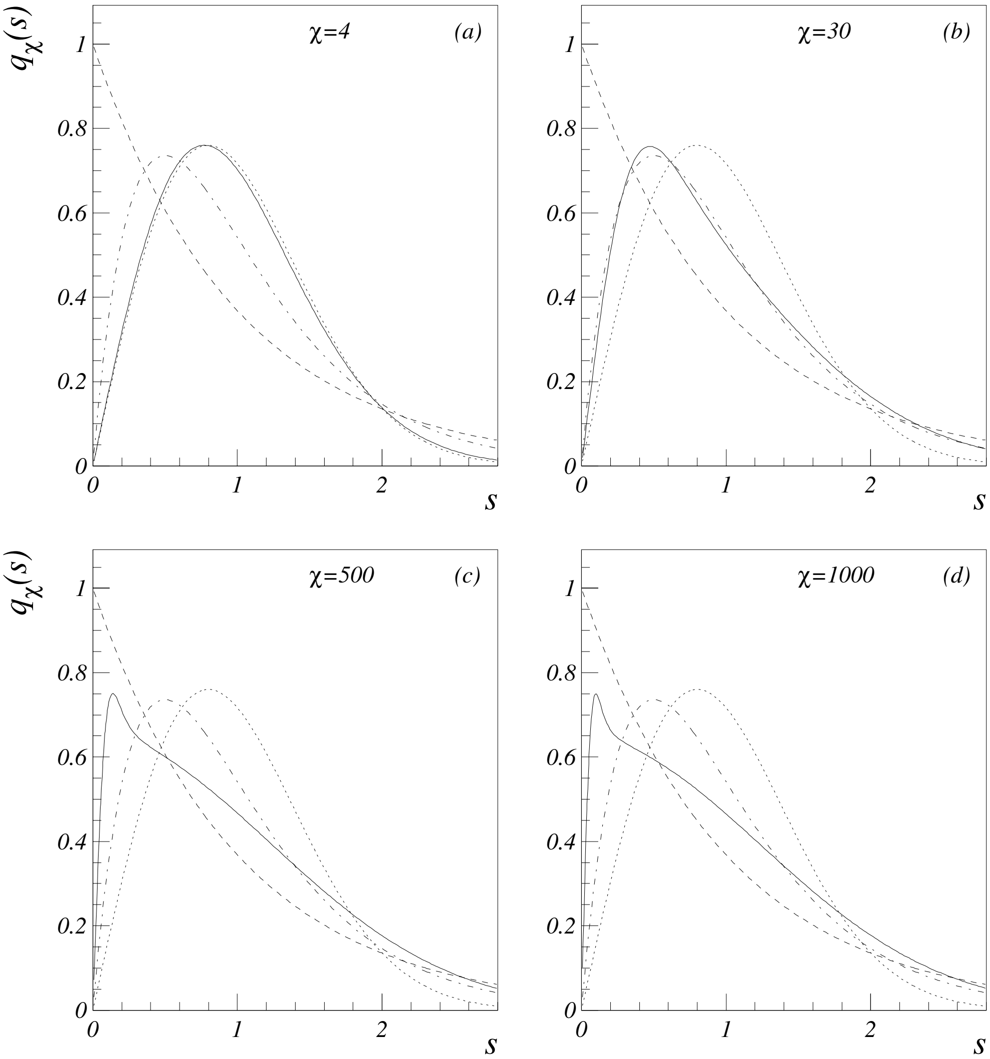

Figure 2. Nearest-neighbour spacing distributions q_χ (solid line) for (*a*) $\chi = 4$, (*b*) $\chi = 30$, (*c*) $\chi = 500$ and (*d*) $\chi = 1000$. The Poisson distribution is plotted as a dashed line, the Wigner distribution as a dotted line and the semi-Poisson as a dashed-dotted line.

This distribution is represented in figure 3 for different values of χ. For $\chi = 2$ it is an exactly uniform distribution, which means that there is no privileged basis (the orthogonal invariance holds). For high values of χ, the initial basis is nearly the eigenbasis (the diagonal elements are much larger than the non-diagonal ones); thus r_χ takes its maximum absolute values for $\theta = 0$ and $\theta = \pi/2$, whereas for small values of χ the eigenstates are more likely obtained after a rotation of $\pi/4$ of the initial basis and r_χ is maximum for $\theta = \pi/4$. For χ and $\chi' = 4/\chi$ the two curves are in quadrature.

We can re-express $p_\chi(E_\alpha, E_\beta)$ as a function of ε and $S = E_\alpha + E_\beta$. Then p_χ can be factorized into a function depending on ε times a function depending on S; i.e., these variables are independent. Moreover, since S is the trace of the matrix it is a Gaussian variable with zero mean and all its odd moments are zero. From the independence of ε and S we deduce

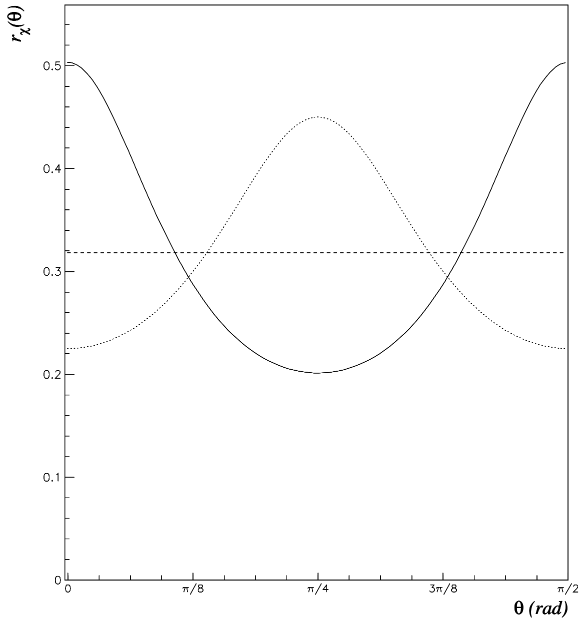

Figure 3. Angular distributions for $\chi = 1$ (dotted line), $\chi = 2$ (dashed line) and $\chi = 5$ (solid line).

that the moments of the eigenvalues fulfil

$$\langle E_\alpha^n \rangle = (-1)^n \langle E_\beta^n \rangle, \tag{17}$$

$$\langle E_\alpha^n \rangle = \frac{1}{2^n} \sum_{p=0}^{n/2} \binom{n}{2p} \langle S^{2p} \rangle \langle \varepsilon^{n-2p} \rangle. \tag{18}$$

From (17) and (18), we can derive the moments of the eigenvalues whose distributions are difficult to compute. Note that we deduce from (17) that the highest and the lowest eigenvalues have opposite mean values and the same variance.

3. Conclusion

The study of the statistical properties of spectra of realistic Hamiltonians requires the consideration of a random matrix ensemble whose elements are not independent or whose distribution is not invariant under orthogonal transformation of a chosen basis. In this letter we have concentrated on the properties of (2×2) real symmetric matrices whose elements are independent Gaussian variables with zero means but do not belong to the GOE. We have derived the distribution of eigenvalues for such a matrix, the NNSD which generalizes the Wigner surmise and we have calculated some important moments. The asymptotic properties of the distribution obtained are functionally identical to those of the ordinary Wigner surmise. For finite χ, the model considered here allows us to describe the transition from chaos to near integrability (the exact integrable limit is never realized). Thus it represents a chaotic system, although with a degree of disorder less important than in the Wigner surmise. The derivation of similar analytical expressions for matrices of larger dimensions is technically difficult. However, we believe that the present results already justify the use of an NNND of type (12) to fit the data as an alternative to the Brody distribution. We also think that these results can be extended to Hermitian matrices.

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